quaderni di matematica
volume 26

edited by
Dipartimento di Matematica
Seconda Università di Napoli

Published with the support of
Seconda Università di Napoli

quaderni di matematica

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Asymmetric Topology and its Applications

edited by Jesús Rodríguez-López and Salvador Romaguera

Received June 2011

Photocomposed copy prepared from a LaTeX file.

ISBN 978-88-548-4752-1

Authors' addresses

S. Cobzaş
Babeş-Bolyai University
Faculty of Mathematics and Computer Science
400 084 Cluj-Napoca, Romania
email: scobzas@math.ubbcluj.ro

Valentín Gregori
Instituto Universitario de Matemática
Pura y Aplicada
Universitat Politècnica de València
46022 Valencia, Spain
email: vgregori@mat.upv.es

Hans-Peter A. Künzi
Department of Mathematics
and Applied Mathematics
University of Cape Town
Rondebosch 7701, South Africa
email: Hans-peter.Kunzi@uct.ac.za

Charly Makitu Kivuvu
Department of Mathematics
and Applied Mathematics
University of Cape Town
Rondebosch 7701, South Africa
email: charly-makitu@yahoo.fr

Jesús Rodríguez-López
Instituto Universitario de Matemática
Pura y Aplicada
Universitat Politècnica de València
46022 Valencia, Spain
email: jrlopez@mat.upv.es

Salvador Romaguera
Instituto Universitario de Matemática
Pura y Aplicada
Universitat Politècnica de València
46022 Valencia, Spain
email: sromague@mat.upv.es

Miguel Ángel Sánchez-Granero
Área de Geometría y Topología
Facultad de Ciencias
Universidad de Almería
04071 Almería, Spain
email: misanche@ual.es

Manuel Sanchis
Departament de Matemàtiques and
Institut de Matemàtiques i
Aplicacions de Castelló
Universitat Jaume I
Campus del Riu Sec. s/n
12071 Castelló, Spain
email: sanchis@mat.uji.es

Almanzor Sapena
Instituto Universitario de Matemática
Pura y Aplicada
Universitat Politècnica de València
46022 Valencia, Spain
email: alsapie@mat.upv.es

Mikhail Tkachenko
Departamento de Matemáticas
Universidad Autónoma Metropolitana
Av. San Rafael Atlixco 186
Col. Vicentina, Del. Iztapalapa
C.P. 09340, México, D.F., Mexico
email: mich@xanum.uam.mx

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quaderni di matematica, 26
ISBN 978-88-548-4752-1
DOI 10.4399/9788854847521
pag. IX-XI

Preface

This special issue of Quaderni di Matematica is devoted to the asymmetric topology. This branch of mathematics studies those topological structures which do not verify a symmetry axiom.

The origins of the theory are closely related to the birth of metrics (Fréchet, 1906). The year before, Pompeiu made use of a metric between sets which was not symmetric, which now is known as the *excess* between two sets. This seems to be the first appearance of a quasi-metric in the literature. Nevertheless, the seminal work of Hausdorff "*Grundzüge der Mengenlehre*" favorated the dissemination of a metric between closed and bounded sets constructed by means of the excess distance of Pompeiu. This metric is the so-called *Hausdorff metric*.

The first results about quasi-metrics were motivated by the metrization theory (Chittenden, 1927; Niemytzki, 1931; Wilson, 1931). In fact, the quasi-metrization problem has become one of the most important research lines of the area and although some solutions have been obtained, people keep on finding a more appealing solution.

The keystone of the asymmetric theory, the quasi-uniformity, was introduced by Nachbin in 1948 when working with topological ordered spaces. This promoted an interest on studying the asymmetric structures (Krishnan, 1955; Császár, 1960; Pervin, 1962; Kelly, 1963; Pervin and Sieber, 1965; Naimpally, 1965; Stoltenberg, 1967, etc.). The first monograph of the area, "*Quasi-uniform topological spaces*", written by Murdeshwar and Naimpally, appeared in 1966. In this book, the basic properties of quasi-uniformities and quasi-metrics are developed, including the Pervin-Sieber completeness and some applications to function spaces.

Closely related to the theory of quasi-uniform spaces, Pervin introduced in 1963 the concept of quasi-proximity. The interplay between quasi-proximities and quasi-uniformities and the important role of these asymmetric structures was set out in the second monograph of the area, "*Quasi-uniform spaces*", due to Fletcher and Lindgren. This book is fundamental for people interested on this subject.

The techniques and methods of the asymmetric topology has been used in several mathematical theories: topological algebra, functional analysis, hyperspaces, fuzzy theory, etc. Moreover this theory has been successfully applied to Computer Science. For example, Smyth (1988) suggested that the most appropriate context for studying language semantics is a quasi-uniform space since it encompasses the advantages of the domains and metric spaces. We also should mention the work of Matthews (1994) who introduced the concept of weightable quasi-metric space (equivalent to partial metric space) as a mathematical model in the study of denotational semantics of data flow networks.

The most recent advances of the asymmetric topology, including its interactions with the areas we have just mentioned, can be consulted in the indispensable works due to Künzi (“*Nonsymmetric distances and their associated topologies: About the origins of basic ideas in the area of asymmetric topology*" (2001), “*An introduction to quasi-uniform spaces*" (2009)).

The papers that we have selected for this special issue pretend to be an introduction to the basic theory of asymmetric topology and to develop the connections with other areas.

The aim of the paper of Cobzaş is to present a survey of some recent results obtained in the study of spaces with asymmetric norm. The presentation follows the ideas from the theory of normed spaces (topology, continuous linear operators, continuous linear functionals, duality, geometry of asymmetric normed spaces, compact operators) emphasizing similarities as well as differences with respect to the classical theory. The main difference comes form the fact that the dual of an asymmetric normed space X is not a linear space, but merely a convex cone in the space of all linear functionals on X. Due to this fact, a careful treatment of the duality problems (e. g. reflexivity) and of other results as, for instance, the extension of fundamental principles of functional analysis (the open mapping theorem and the closed graph theorem) to this setting, is needed.

One of the main intricate theories of the asymmetric topology is the completion theory. In the literature there exist several different completion notions but all of them give rise to the classical notion of completeness when working on the symmetric context. The following two papers are devoted to this theory.
Gregori, Romaguera and Sapena study the completion theory for fuzzy quasi-metric spaces in the sense of George and Veeramani. They consider two different theories of fuzzy quasi-metric completion which are shown to be appropriate. The first one is called the fuzzy quasi-metric bicompletion and constitutes the fuzzy counterpart of the bicompletion of quasi-metric spaces. The second one is developed under the name of FD-completion and constitutes the fuzzy counterpart of Doitchinov completion of quasi-metric spaces.
In their article, Künzi and Makitu Kivivu try to extend the theory of the Doitchinov completion of quiet T_0-quasi-uniform spaces to general T_0-quasi-uniform spaces. Unfortunately, it seems that this theory cannot be generalized, which establishes an important difference between the D-completion theory for

balanced T_0-quasi-metric spaces and quiet T_0-quasi-uniform spaces. Therefore, the authors suggest to work, in an arbitrary quasi-uniform space, with a chosen nonempty subbasic family of quasi-pseudometrics and a concept of balancedness with respect to that family. In this way they obtain a general "quasi-pseudometric" theory of the B-completion for subbasic T_0-families of quasi-pseudometrics that can be applied to the study of quasi-uniform spaces.

The paper due to Rodríguez-López contains a brief introduction to the theory of quasi-uniform spaces and their interactions with quasi-metrics and quasi-proximities. It stresses the differences between the classical theory and the asymmetric one.

On the other hand, Sánchez-Granero gives a survey of the so-called fractal structures and their relation with transitive quasi-uniformities and ultra-quasimetrics. He shows that the concept of fractal structure is natural in some different contexts, like metrization, fractal dimension, transitive quasi-uniformities and self similar sets. These fractal structures allow to characterize topological and quasi-uniform properties as, for example, compactness, connectedness, paracompactness, completeness, etc. Some other uses of fractal structures are also commented like, for example, how to use them to construct curves or continuous maps and how they can be used to study continua and compactifications.

Finally, Sanchis and Tkachenko present a concise survey of results on semi-topological and paratopological groups that have mainly been obtained during the last decade. Primarily they focuss attention on cardinal invariants and embedding and condensation problems. Then they consider paratopological groups as topologically asymmetric objects and study their properties from the bitopological point of view. Recent advances in free paratopological groups are also presented.

Jesús Rodríguez-López and Salvador Romaguera

quaderni di matematica, 26
ISBN 978-88-548-4752-1
DOI 10.4399/9788854847522
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Functional analysis in asymmetric normed spaces

S. Cobzaş

Contents

1. Introduction

The aim of this paper is to present the basic results on the so called asymmetric normed spaces, by analogy with the classical theory of normed linear spaces. An asymmetric norm is a positive sublinear functional p defined on a real linear space X such that $p(x) = p(-x) = 0$ implies $x = 0$. This functional can be obtained as the Minkowski gauge functional of an absorbing convex subset, and the possibility $p(x) \neq p(-x)$ is not excluded, explaining the adjective "asymmetric". Asymmetric metric spaces are called quasi-metric spaces. It is difficult to localize the first moment when asymmetric norms were used, but it goes back as early as 1968 in a paper by Duffin and Karlovitz [50], who proposed the term asymmetric norm. Krein and Nudelman [94] used also asymmetric norms in their study of some extremal problems related to the Markov moment problem. Remark that the relevance of sublinear functionals for some problems of convex analysis and of mathematical analysis was emphasized also by H. König, see [86, 87, 89] and the survey paper [88]. But a systematic study of the properties of asymmetric normed spaces started with the papers of S. Romaguera, from the Polytechnic University of Valencia, and his collaborators from the same university and from other universities in Spain: Alegre, Ferrer, García-Raffi, Sánchez Pérez, Sánchez Álvarez, Sanchis, Valero (see the bibliography at the end of this paper). Beside its intrinsic interest, their study was motivated also by the applications in Computer Science, namely to the complexity analysis of programs, results obtained in cooperation with Professor Schellekens from National University of Ireland.

A general idea about the topics included in this survey can be obtained from the above contents. Due to the fact that completeness and compactness play a central role in functional analysis, we emphasize in the second section some of the difficulties arising in studying the relations between completeness, compactness, total boundedness and precompactness within the framework of quasi-metric and quasi-uniform spaces.

A word must be said about notation. We denote by $\mathbb{N} = \{1, 2, \dots\}$ the set of natural numbers (positive integers). Intervals are denoted by $[a; b]$, $(a; b)$, $(a; b]$, $[a; b)$, while the notation (a, b) is used to designate an ordered pair. A closed ball

*This research was supported by Grant CNCSIS 2261,Contract 543/2008.

in a quasi-metric space (X,ρ) is denoted by $B_\rho[x,r]=\{y\in X:\rho(x,y)\le r\}$, and an open ball by $B_\rho(x,r)=\{y\in X:\rho(x,y)<r\}$. The closed unit ball of an asymmetric normed space (X,p) is denoted by B_p, the open unit ball by B'_p and the unit sphere by S_p. The rest of the notation is standard or explained in the text.

1.1. Quasi-metric spaces and asymmetric normed spaces

A *quasi-semimetric* on an arbitrary set X is a mapping $\rho : X\times X\to[0;\infty)$ satisfying the following conditions:

$$\text{(QM1)}\qquad \rho(x,y)\ge 0, \quad \textit{and} \quad \rho(x,x)=0;$$
$$\text{(QM2)}\qquad \rho(x,z)\le\rho(x,y)+\rho(y,z),$$

for all $x,y,z\in X$. If, further,

$$\text{(QM3)}\qquad \rho(x,y)=\rho(y,x)=0\Rightarrow x=y,$$

for all $x,y\in X$, then ρ is called a *quasi-metric*. The pair (X,ρ) is called a *quasi-semimetric space*, respectively a *quasi-metric space*. The conjugate of the quasi-semimetric ρ is the quasi-semimetric $\bar\rho(x,y)=\rho(y,x)$, $x,y\in X$. The mapping $\rho^s(x,y)=\max\{\rho(x,y),\bar\rho(x,y)\}$, $x,y\in X$, is a semimetric on X which is a metric if and only if ρ is a quasi-metric. Sometimes one works with *extended* quasi-semimetrics, meaning that the quasi-semimetric ρ can take the value $+\infty$ for some $x,y\in X$. The following inequalities hold for these quasi-semimetrics for all $x,y\in X$:

$$\rho(x,y)\le\rho^s(x,y)\quad\text{and}\quad\bar\rho(x,y)\le\rho^s(x,y). \tag{1.1}$$

An *asymmetric norm* on a real vector space X is a functional $p:X\to[0,\infty)$ satisfying the conditions

$$\text{(AN1) } p(x)=p(-x)=0\Rightarrow x=0;\quad \text{(AN2) } p(\alpha x)=\alpha p(x);$$
$$\text{(AN3) } p(x+y)\le p(x)+p(y),$$

for all $x,y\in X$ and $\alpha\ge 0$. If p satisfies only the conditions (AN2) and (AN3), then it is called an *asymmetric seminorm*. The pair (X,p) is called an *asymmetric normed* (respectively *seminormed*) *space*. Again, in some instances, the value $+\infty$ will be allowed for p in which case we shall call it an *extended asymmetric norm* (or seminorm). An asymmetric seminorm p defines a quasi-semimetric ρ_p on X through the formula

$$\rho_p(x,y) = p(y-x),\ x,y \in X. \tag{1.2}$$

Defining the conjugate asymmetric seminorm $\bar{p}$ and the seminorm p^s by

$$\bar{p}(x) = p(-x) \quad \text{and} \quad p^s(x) = \max\{p(x), p(-x)\} \tag{1.3}$$

for $x \in X$, the inequalities (1.1) become

$$p(x) \le p^s(x) \quad \text{and} \quad \bar{p}(x) \le p^s(x), \tag{1.4}$$

for all $x \in X$. Obviously, p^s is a norm when p is an asymmetric norm.

The conjugates of ρ and p are denoted also by ρ^{-1} and p^{-1}, a notation that we shall use occasionally.

If (X,ρ) is a quasi-semimetric space, then for $x \in X$ and $r > 0$ we define the balls in X by the formulae

$$\begin{aligned} B_\rho(x,r) =& \{y \in X : \rho(x,y) < r\} \text{ - the open ball, and} \\ B_\rho[x,r] =& \{y \in X : \rho(x,y) \le r\} \text{ - the closed ball.} \end{aligned}$$

In the case of an asymmetric seminormed space (X,p) the balls are given by

$$\begin{aligned} B_p(x,r) &= \{y \in X : p(y-x) < r\}, \text{ respectively} \\ B_p[x,r] &= \{y \in X : p(y-x) \le r\}. \end{aligned}$$

The closed unit ball of X is $B_p = B_p[0,1]$ and the open unit ball is $B'_p = B_p(0,1)$. In this case the following formulae hold true

$$B_p[x,r] = x + rB_p \quad \text{and} \quad B_p(x,r) = x + rB'_p, \tag{1.5}$$

that is, any of the unit balls of X completely determines its metric structure. If necessary, these balls will be denoted by $B_{p,X}$ and $B'_{p,X}$, respectively.

The conjugate $\bar{p}$ of p is defined by $p(x) = p(-x)$, $x \in X$, and the associate seminorm is $p^s(x) = \max\{p(x), \bar{p}(x)\}$, $x \in X$. The seminorm p is an asymmetric norm if and only if p^s is a norm on X. Sometimes an asymmetric norm will be denoted by the symbol $\|\cdot|$, a notation proposed by Krein and Nudelman, [94, Ch. IX, §5], in their book on the theory of moments.

Remark 1.1 - Since the terms "quasi-norm", "quasi-normed space" and "quasi-Banach space" are already "registered trademarks" (see, for instance, the survey by Kalton [77]), we can not use these terms to designate an asymmetric norm, an asymmetric normed space or an asymmetric biBanach space. A quasi-normed space is a vector space X equipped with a functional $\|\cdot\| : X \to [0;\infty)$, satisfying all the axioms of a norm, excepting the triangle inequality which is replaced by:

$$\|x+y\| \le C(\|x\| + \|y\|),\ x, y \in X,$$

for some constant $C \ge 1$. It is obvious that for $C = 1$ the functional $\|\cdot\|$ is a norm. The reverse situation is also encountered: in [169] a quasi-metric space is a metric space (X, ρ) in which the triangle inequality is replaced by $\rho(x,z) \le C\left(\rho(x,y) + \rho(y,z)\right)$, for some $C > 0$.

1.2. The topology of a quasi-semimetric space

The topology $\tau(\rho)$ of a quasi-semimetric space (X, ρ) can be defined starting from the family $\mathcal{V}_\rho(x)$ of neighborhoods of an arbitrary point $x \in X$:

$$\begin{aligned} V \in \mathcal{V}_\rho(x) &\iff \exists r > 0 \text{ such that } B_\rho(x,r) \subset V \\ &\iff \exists r' > 0 \text{ such that } B_\rho[x,r'] \subset V. \end{aligned}$$

To see the equivalence in the above definition, we can take, for instance, $r' = r/2$.

A set $G \subset X$ is $\tau(\rho)$-open if and only if for every $x \in G$ there exists $r = r_x > 0$ such that $B_\rho(x,r) \subset G$. Sometimes we shall say that V is a ρ-neighborhood of x or that the set G is ρ-open.

The convergence of a sequence (x_n) to x with respect to $\tau(\rho)$, called ρ-convergence and denoted by $x_n \xrightarrow{\rho} x$, can be characterized in the following way

$$x_n \xrightarrow{\rho} x \iff \rho(x, x_n) \to 0. \tag{1.6}$$

Also

$$x_n \xrightarrow{\bar{\rho}} x \iff \bar{\rho}(x, x_n) \to 0 \iff \rho(x_n, x) \to 0. \tag{1.7}$$

Using the conjugate quasi-semimetric $\bar{\rho}$ one obtains another topology $\tau(\bar{\rho})$. A third one is the topology $\tau(\rho^s)$ generated by the semimetric ρ^s. Sometimes, (see, for instance, Menucci [108] and Collins and Zimmer [36]) the balls with respect to ρ are called *forward balls* and the topology $\tau(\rho)$ is called the *forward topology*, while the balls with respect to $\bar{\rho}$ are called *backward balls* and the topology $\tau(\bar{\rho})$ the *backward topology*. We shall use sometimes the alternative notation τ_ρ, $\tau_{\bar{\rho}}$, τ_{ρ^s} to designate these topologies.

As a space with two topologies, τ_ρ and $\tau_{\bar{\rho}}$, a quasi-semimetric space can be viewed as a bitopological space in the sense of Kelly [79] (see also the book [52]) and so, all the results valid for bitopological spaces apply to a quasi-semimetric space. A *bitopological space* is simply a set T endowed with two topologies τ and σ. A bitopological space is denoted by (T, τ, σ).

The following example is very important in what follows.

Example 1.2 - On the field $\mathbb{R}$ of real numbers consider the asymmetric norm $u(\alpha) = \alpha^+ := \max\{\alpha, 0\}$. Then, for $\alpha \in \mathbb{R}$, $\bar{u}(\alpha) = \alpha^- := \max\{-\alpha, 0\}$ and $u^s(\alpha) = |\alpha|$. The topology $\tau(u)$ generated by u is called the *upper topology* of $\mathbb{R}$, while the topology $\tau(\bar{u})$ generated by $\bar{u}$ is called the *lower topology* of $\mathbb{R}$. A basis of $\tau(u)$-neighborhoods of a point $\alpha \in \mathbb{R}$ is formed of the intervals $(-\infty; \alpha + \epsilon)$, $\epsilon > 0$. A basis of $\tau(\bar{u})$-neighborhoods is formed of the intervals $(\alpha - \epsilon; \infty)$, $\epsilon > 0$.

In this space the addition is continuous from $(\mathbb{R} \times \mathbb{R}, \tau_u \times \tau_u)$ to $(\mathbb{R}, \tau_u)$, but the multiplication is not continuous at any point $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$.

The continuity property can be directly verified. To see the last assertion, let $V = (-\infty; \alpha\beta + \epsilon)$, be a τ_u-neighborhood of $\alpha\beta$, for some $\epsilon > 0$. Since the τ_u-neighborhoods of α and β contain $-n$, for $n \in \mathbb{N}$ sufficiently large, it follows that $n^2 = (-n)(-n)$ does not belong to V, for n large enough.

Other important topological example is the so called Sorgenfrey topology on $\mathbb{R}$.

Example 1.3 - [The Sorgenfrey line] For $x, y \in \mathbb{R}$ define a quasi-metric ρ by $\rho(x, y) = y - x,$ if $x \leq y$ and $\rho(x, y) = 1$ if $x > y$. A basis of τ_ρ-neighborhoods of a point $x \in \mathbb{R}$ is formed by the family $[x; x + \epsilon)$, $0 < \epsilon < 1$. The family of intervals $(x - \epsilon; x]$, $0 < \epsilon < 1$, forms a basis of $\tau_{\bar{\rho}}$-neighborhoods of x. Obviously, the topologies τ_ρ and $\tau_{\bar{\rho}}$ are Hausdorff and $\rho^s(x, y) = 1$ for $x \neq y$, so that $\tau(\rho^s)$ is the discrete topology of $\mathbb{R}$.

We shall present, for the convenience of the reader, the separation axioms. A topological space (T, τ) is called

- T_0 if for any pair s, t of distinct points in T, at least one of them has a neighborhood not containing the other;

- T_1 if for any pair s, t of distinct points in T, each of them has a neighborhood not containing the other (this is equivalent to the fact that the set $\{t\}$ is closed for every $t \in T$);

- *Hausdorff* or T_2 if for any pair s, t of distinct points in T, there exist neighborhoods U of s and V of t such that $U \cap V = \emptyset$;

- *regular* if for each $t \in T$ and each closed subset S of T, not containing t, there are disjoint open subsets U, V of T such that $t \in U$ and $S \subset V$. In other words a point and a closed set not containing it can be separated by open sets. This is equivalent to the fact that every point in T has a neighborhood base formed of closed sets. If T is regular and T_1, then it is called a T_3 space.

- *completely regular*, or *Tychonoff*, or $T_{3\frac{1}{2}}$, if for every $t \in T$ and every closed subset S of T not containing t there is a continuous function $f : T \to [0; 1]$ such that $f(t) = 1$ and $f(s) = 0$ for each $s \in S$.

- *normal* if any pair S_1, S_2 of disjoint closed sets can be separated by open sets, that is there exist two disjoint open sets $G_1 \supset S_1$ and $G_2 \supset S_2$. A normal T_1 space is called a T_4 space.

We introduce, following Kelly [79], some separation properties specific to a bitopological space (T, τ, σ). The bitopological space (T, τ, σ) is called *pairwise Hausdorff* if for each pair of distinct points $s, t \in T$ there exists a τ-neighborhood U of s and a σ-neighborhood V of t such that $U \cap V = \emptyset$. It is obvious that if T is pairwise Hausdorff, then each of the topologies τ and σ are T_1. The topology τ is called *regular with respect to* σ if every $t \in T$ has a τ-neighborhood base formed of σ-closed sets or, equivalently, if for every $t \in T$ and every τ-closed subset S of T not containing t, there exist a τ-open set U and a σ-open set V such that $t \in U$, $S \subset V$ and $U \cap V = \emptyset$. The bitopological space (T, τ, σ) is called *pairwise regular* if τ is regular with respect to σ and σ is regular with respect to τ. The bitopological space (T, τ, σ) is called *pairwise normal* if given a τ-closed subset A of T and a σ-closed subset B of T with $A \cap B = \emptyset$, there exist a σ-open subset U of T and a τ-open subset V of T such that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$. Using these notions, Kelly proved in [79] some extension and existence results for semicontinuous functions similar to the classical theorems of Tietze and Uryson. Note also the following result from Kelly [79].

Theorem 1.4. *If (T, τ, σ) is a pairwise regular bitopological space such that both τ and σ satisfy the second axiom of countability, then it is quasi-semimetrizable. If further, T is pairwise Hausdorff, then it is quasi-metrizable.*

A bitopological space (T, τ, σ) is called *quasi-semimetrizable* if there exists a quasi-semimetric ρ on T such that $\tau = \tau_\rho$ and $\sigma = \tau_{\bar{\rho}}$. If ρ is a semimetric, then $\tau = \sigma$. The following topological properties are true for quasi-semimetric spaces.

Proposition 1.5. *If (X, ρ) is a quasi-semimetric space, then*

1. *Any ball $B_\rho(x, r)$ is $\tau(\rho)$-open and a ball $B_\rho[x, r]$ is $\tau(\bar{\rho})$-closed. The ball $B_\rho[x, r]$ need not be $\tau(\rho)$-closed.*

   Also, the following inclusions hold

$$B_{\rho^s}(x, r) \subset B_\rho(x, r) \text{ and } B_{\rho^s}(x, r) \subset B_{\bar{\rho}}(x, r),$$

   with similar inclusions for the closed balls.